

Brans-Dicke Gravity from Entropic Viewpoint

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We interpret the Brans-Dicke gravity from entropic viewpoint. We first apply the Verlinde's entropic formalism in the Einstein frame, then perform the conformal transformation which connects the Einstein frame to the Jordan frame. The transformed result yields the equation of motion of the Brans-Dicke theory in the Jordan frame.

I. INTRODUCTION

It is well-known that a black hole has an entropy [1], and a black hole radiates as if it has a temperature proportional to the surface gravity on its event horizon [2]. These facts suggest that there may exist a deep connection between gravity and thermodynamics. Then, Jacobson [3] showed that the Einstein equation can be obtained from the first law of thermodynamics together with the Bekenstein-Hawking entropy-area relation [1, 2] applied on a local Rindler horizon [4].

Recently, Verlinde [5] proposed that gravity is not a fundamental interaction but can be explained as a macroscopic thermodynamic phenomenon. Based on the holographic principle and the equipartition rule, he showed that gravity emerges as an entropic force. A related idea was considered also by Padmanabhan [6, 7]. Since these works, there have appeared many subsequent investigations in cosmology [8], black hole physics [9–12], loop quantum gravity [13], and other fields.

In this paper we investigate whether the Brans-Dicke gravity can be viewed as an entropic phenomenon à la Verlinde. The Brans-Dicke gravity is characterized by the fact that the

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gravitational coupling is not presumed to be a constant but is proportional to the inverse of a scalar field which couples non-minimally to the curvature scalar [14]. Also, it is well known that Brans-Dicke theory can be transformed the Einstein frame via a conformal transformation, and in the Einstein frame Brans-Dicke action is equivalent to the Einstein gravity plus the scalar field action except the fact that unlike in the Jordan frame a usual matter does not follow the geodesic of Einstein frame [14]. We use the holographic principle and the equipartition rule in the Einstein frame first, and then via a conformal transformation we recover the field equations of metric in the Jordan frame.

In Ref. [15] it was shown that black hole entropy in the Brans-Dicke gravity depends upon the value of scalar field at the horizon as well as the area of horizon. However, we will show that a naive application of Verlinde's formalism with this entropy expression of the Jordan frame cannot yield the correct field equations.

The paper is organized as follows. First we review the Verlinde's conjecture on the emergent gravity in the case of the Einstein gravity, then give an entropic derivation of the Brans-Dicke field equation based on the holographic principle and the equipartition rule. We work with $c = \hbar = k_B = 1$ in this paper.

II. EINSTEIN EQUATIONS À LA VERLINDE

According to Verlinde, gravity is an entropic force emerging from coarse graining process of information for a given energy distribution. In this process, information is stored on the holographic screen.

In a static background with a global timelike Killing vector ξ^a , a generalized Newtonian potential is defined by

$$\Phi = \log \sqrt{-\xi^a \xi_a}. \quad (1)$$

With this potential we can foliate the spacetime, and we choose the holographic screen as an equipotential surface of Φ . In this background, a test particle approaching the holographic screen experiences the following 4-acceleration

$$a^a = u^b \nabla_b u^a = -\nabla^a \Phi \quad (2)$$

where u^a is the 4-velocity of the particle. The temperature of the holographic screen measured by an observer at infinity is given by the so-called Unruh-Verlinde temperature which

is obtained by multiplying the redshift factor $e^\Phi = \sqrt{-\xi^a \xi_a}$ to the Unruh temperature for the acceleration (2):

$$T = \frac{1}{2\pi} e^\Phi N^a \nabla_a \Phi, \quad (3)$$

where N^a is the outward unit normal vector to the holographic screen and the Killing vector. With the holographic principle which states that the information in the bulk can be given by the information on the boundary, and with the equipartition rule which states that each bit of information contributes the energy of $\frac{1}{2}T$, the quasi-local energy inside the holographic screen which is the boundary of spacelike hypersurface Σ can be written as

$$E = \oint_{\partial\Sigma} \frac{T}{2} dN = \frac{1}{4\pi G} \oint_{\partial\Sigma} e^\Phi N^b \nabla_b \Phi dA, \quad (4)$$

where the number of bits on an area dA is assumed to be $dN = dA/G$. With (1) and the properties of Killing vector ξ^a , one can show that the right hand side of (4) is nothing but the Komar expression

$$E_{\text{Komar}} = \frac{1}{4\pi G} \oint_{\partial\Sigma} dA_{ab} \nabla^{[a} \xi^{b]}, \quad (5)$$

where $dA_{ab} = \frac{1}{2!} \epsilon_{abcd} dx^c \wedge dx^d$. Expressing the Komar mass inside the holographic screen with the energy momentum tensor T_{ab} in the left-hand side and applying the Stokes theorem to the right-hand side of Eq. (5) together with an identity, $\nabla^a \nabla_a \xi^b = -R^{ab} \xi_a$ we have [16]

$$2 \int_{\Sigma} \left(T_{ab} - \frac{1}{2} g_{ab} T^c_c \right) \xi^b d\Sigma^a = \frac{1}{4\pi G} \int_{\Sigma} R_{ab} \xi^b d\Sigma^a, \quad (6)$$

where $d\Sigma^a$ is defined with an outward pointing normal. Since Eq. (6) is true for an arbitrary holographic screen, we obtain the Einstein equation

$$R_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2} g_{ab} T^c_c \right). \quad (7)$$

III. BRANS-DICKE EQUATIONS FROM ENTROPIC VIEWPOINT

In the previous section we have seen that Einstein gravity can be seen as an entropic phenomenon. In this section we will extend this entropic viewpoint of gravity to the Brans-Dicke theory. In the Brans-Dicke theory there are two frames called the Jordan frame and the Einstein frame whose metrics we denote by \tilde{g}_{ab} and g_{ab} , respectively. These two frames are related by the following conformal transformation:

$$g_{ab} = \psi \tilde{g}_{ab}. \quad (8)$$

Unlike the minimal coupling in the Einstein frame, the scalar field ψ couples non-minimally to the curvature scalar in the Jordan frame such that it plays the role of effective gravitational coupling of the theory, i.e., $G_{\text{eff}} = G/\psi$. Since this extra scalar field comes into play, the gravity in the Jordan frame is also known as the scalar-tensor theory of gravity. From now on, all quantities with tilde should be understood as expressions in the Jordan frame.

In the following we will show that the Brans-Dicke field equation of metric in the Jordan frame can be obtained with Verlinde's entropic formulation. Since it is well known that the entropy of stationary black hole in the Jordan frame is proved to be the same as that in the Einstein frame [17], first, we will setup entropic formulation in the Einstein frame where the usual Verlinde's idea works nicely for the Einstein gravity and then with a conformal transformation (8) we will derive the field equations of \tilde{g}_{ab} in the Jordan frame.

As usual we assume that there exists a timelike killing vector $\tilde{\xi}^a$ in the Jordan frame satisfying the Killing equation, $\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a = 0$ where $\tilde{\nabla}_a$ is the covariant derivative associated with the Jordan metric \tilde{g}_{ab} . Notice that $\xi_a = g_{ab} \tilde{\xi}^b$ satisfies the Killing condition in the Einstein frame.

The gravitational potential $\Phi = \log \sqrt{-\xi^a \xi_a}$ associated with ξ^a in the Einstein frame has the following relation with the gravitational potential $\tilde{\Phi}$ in the Jordan frame:

$$\Phi = \tilde{\Phi} + \frac{1}{2} \log \psi, \quad (9)$$

where $\tilde{\Phi} = \log \sqrt{-\tilde{\xi}^a \tilde{\xi}_a}$. Therefore the acceleration of a test particle near the holographic screen seen in the Einstein frame is given by

$$a_a = -\nabla_a \Phi = \tilde{a}_a - \frac{1}{2} \tilde{\nabla}_a \log \psi, \quad (10)$$

where the acceleration measured in the Jordan frame is given by $\tilde{a}_a = -\tilde{\nabla}_a \tilde{\Phi}$. Notice that unlike the Jordan frame description of the Brans-Dicke gravity, $a_a = 0$ does not mean the geodesic motion of a test particle in the Einstein frame. Eq. (10) shows that a measure of free-falling in the Einstein frame is given by $a_a + \frac{1}{2} \nabla_a \psi$ which compensates for the additional scalar field dragging.

With this acceleration a_a , we define the Unruh-Verlinde temperature T in the Einstein frame as follows:

$$T = e^\Phi N^a a_a = e^{\tilde{\Phi}} \tilde{N}^a \left(\tilde{a}_a + \frac{1}{2} \tilde{\nabla}_a \log \psi \right), \quad (11)$$

where N_a and \tilde{N}_a are the unit outward normal vectors to the holographic screen seen in the Einstein frame and in the Jordan frame, respectively, and are related by $N_a = \sqrt{\psi} \tilde{N}_a$. Therefore the quasi-local energy (4) inside the holographic screen $\partial\Sigma$ seen in the Einstein frame has the following expression in terms of variables in the Jordan frame:

$$E = \oint_{\partial\Sigma} \frac{T}{2} \frac{dA}{G} = \frac{1}{4\pi G} \oint_{\partial\tilde{\Sigma}=\partial\Sigma} e^{\tilde{\Phi}} \tilde{N}^a \left(\tilde{a}_a + \frac{1}{2} \tilde{\nabla}_a \log \psi \right) \psi d\tilde{A}, \quad (12)$$

where the surface element is given by $dA = \psi d\tilde{A}$ under the conformal transformation (8). Here $\partial\tilde{\Sigma}$ is the same closed hypersurface as $\partial\Sigma$ but seen in the Jordan frame. The same applies to $\tilde{\Sigma}$ and Σ below. Now, applying the Stokes theorem to each terms in the right-hand side of Eq. (12) yields

$$\frac{1}{4\pi G} \oint_{\partial\tilde{\Sigma}} e^{\tilde{\Phi}} \tilde{N}^a \tilde{a}_a \psi d\tilde{A} = \frac{1}{4\pi G} \oint_{\partial\tilde{\Sigma}} \psi \tilde{\nabla}^{[a} \tilde{\xi}^{b]} d\tilde{A}_{ab} = \frac{1}{4\pi G} \int_{\tilde{\Sigma}=\Sigma} \left(\psi \tilde{R}_{ab} - \tilde{\nabla}_a \tilde{\nabla}_b \psi \right) \tilde{\xi}^b d\tilde{\Sigma}^a, \quad (13)$$

and

$$\frac{1}{8\pi G} \oint_{\partial\tilde{\Sigma}} e^{\tilde{\Phi}} \tilde{N}^a (\tilde{\nabla}_a \log \psi) \psi d\tilde{A} = -\frac{1}{8\pi G} \oint_{\partial\tilde{\Sigma}} \tilde{\xi}^{[a} \tilde{\nabla}^{b]} \psi d\tilde{A}_{ab} = -\frac{1}{8\pi G} \int_{\tilde{\Sigma}=\Sigma} \tilde{g}_{ab} \tilde{\square} \psi \tilde{\xi}^b d\tilde{\Sigma}^a, \quad (14)$$

where $d\tilde{\Sigma}^a$ is the volume measure in the Jordan frame. Thus, we have

$$E = \frac{1}{4\pi G} \int_{\tilde{\Sigma}} \left[\psi \tilde{R}_{ab} - \left(\tilde{\nabla}_a \tilde{\nabla}_b + \frac{1}{2} \tilde{g}_{ab} \tilde{\square} \right) \psi \right] \tilde{\xi}^b d\tilde{\Sigma}^a, \quad (15)$$

where $\tilde{\square} = \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b$. As we have done in Eq. (6) we express E in the terms of energy-momentum tensor as follows:

$$E = 2 \int_{\Sigma} \left(T_{ab} - \frac{1}{2} g_{ab} T^c_c \right) \xi^b d\Sigma^a = 2 \int_{\tilde{\Sigma}=\Sigma} \left(\tilde{T}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{T}^c_c \right) \tilde{\xi}^b d\tilde{\Sigma}^a. \quad (16)$$

Here, we have used the fact that the energy-momentum tensor in the Einstein frame has the following relation with that in the Jordan frame under the conformal transformation (8),

$$T_{ab} = \psi^{-1} \tilde{T}_{ab}. \quad (17)$$

We emphasize that the energy momentum tensor in (16) contains the contribution from scalar field ψ as well as the ordinary matters.

Now, comparing two expressions (15) and (16) for the holographic energy we get the field equations for metric \tilde{g}_{ab} as follows:

$$\tilde{R}_{ab} = \frac{8\pi G}{\psi} \left(\tilde{T}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{T}^c_c \right) + \frac{1}{\psi} \left(\tilde{\nabla}_a \tilde{\nabla}_b + \frac{1}{2} \tilde{g}_{ab} \tilde{\square} \right) \psi. \quad (18)$$

This equation is the same one which can be obtained by varying the Brans-Dicke action with respect to the Jordan metric \tilde{g}_{ab} :

$$I_{\text{BD}} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \psi \tilde{R} + I[\tilde{g}_{ab}, \psi] + I_{\text{matter}}[\tilde{g}_{ab}, \phi^{(\text{m})}], \quad (19)$$

where $I[\tilde{g}_{ab}, \psi]$ is the action for scalar field ψ and $I_{\text{matter}}[\tilde{g}_{ab}, \phi^{(\text{m})}]$ is an ordinary matter action minimally coupled to \tilde{g}_{ab} . Variation of last two terms in (19) with respect to \tilde{g}_{ab} will contribute to the energy momentum tensor of (18). Since the entropic consideration is restricted to the gravity sector, we cannot give explicit form of $I[\tilde{g}_{ab}, \psi]$ in (19) within our formulation. However, the coupling $\psi \tilde{R}$ in (19) manifests the fact that the model we are considering is the Brans-Dicke gravity. A prototype of the scalar field action in the Brans-Dicke is usually given by

$$I[\tilde{g}_{ab}, \psi] = \frac{1}{16\pi G} \int d^4x \sqrt{\tilde{g}} \frac{\omega}{\psi} \tilde{g}^{ab} \tilde{\nabla}_a \psi \tilde{\nabla}_b \psi, \quad (20)$$

where ω is the Brans-Dicke parameter.

With the holographic principle and the equipartition rule we have derived the field equations for metric of the Brans-Dicke gravity without resort to the action principle as Verlinde has done for the Einstein gravity.

Finally, we comment that naive application of the holographic principle and the equipartition rule in the Jordan frame does not yield the correct field equations of Jordan metric. Since in the Brans-Dicke gravity the entropy of black hole with horizon area \tilde{A} is given by [15]

$$\tilde{S}_{\text{BH}} = \frac{\psi \tilde{A}}{4G} \Big|_{\text{horizon}}, \quad (21)$$

we expect that an infinitesimal area $d\tilde{A}$ on the holographic screen measured in the Jordan frame would contain the following amount of information

$$d\tilde{N} = \frac{d\tilde{A}}{G_{\text{eff}}} = \frac{\psi d\tilde{A}}{G}. \quad (22)$$

Thus one may write the quasi-local energy inside the holographic screen $\partial\tilde{\Sigma}$ of Jordan frame which is compatible with the holographic principle and the equipartition rule as follows

$$\tilde{E} = \oint_{\partial\tilde{\Sigma}} \frac{\tilde{T}}{2} \frac{\psi d\tilde{A}}{G}. \quad (23)$$

It is obvious from the Eq. (12) that in order to obtain the Brans-Dicke field equations the temperature \tilde{T} in this expression should be equal to the Unruh-Verlinde temperature not of Jordan frame but of Einstein frame in (11).

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